REPRESENTATION THEORY OF COMBINATORIAL CATEGORIES

JOHN WILTSHIRE-GORDON
(PORTIONS ARE JOINT WITH JORDAN ELLENBERG)

Thanks to Vic Reiner and Jed Yang
for the invitation.

Representation Theory of Combinatorial Categories

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6. Detecting Artinian categories

Representation Theory of Combinatorial Categories

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2. What?
3. Where?
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5. Interlude on the Dold-Kan Correspondence
6. Detecting Artinian categories
7. Example and consequences.

1

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$$\sigma \longmapsto M_\sigma$$

rules for converting permutations to square matrices

①

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$$M_e = I$$

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$$M_f \circ M_g = M_{f \circ g}$$

$$M_{\mathbb{1}} = I.$$

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we have an induced map

$$Mf : M[m] \longrightarrow M[n]$$

so that $$(Mf) \circ (Mg) = M(f \circ g)$$

and $$M\mathbb{1}_{[n]} = \mathbb{1}_{M[n]}$$

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In other words, M is a functor

$$M : \mathcal{F} \longrightarrow \mathrm{Vect}_{\mathbb{F}}$$

where $\mathcal{F}$ is the category whose objects are $[n] = \{1, \dots, n\}$ and whose arrows are functions so that $(Mf) \circ (Mg) = M(f \circ g)$

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$\mathcal{F}$ is the primordial "combinatorial category".

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and so the entire category $\mathcal{F}$ acts, not just a single symmetric group.

$$V[n] = \left\langle x_i \otimes x_j \otimes x_k \otimes x_\ell \right\rangle$$

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$$V[n] = \frac{\left\langle x_i \otimes x_j \otimes x_k \otimes x_\ell \right\rangle}{\left\langle \begin{array}{l} 23 x_i \otimes x_j \otimes x_k \otimes x_\ell + x_j \otimes x_i \otimes x_k \otimes x_\ell - x_j \otimes x_k \otimes x_i \otimes x_\ell + \cdots - x_\ell \otimes x_k \otimes x_j \otimes x_i , \\ \quad x_i \otimes x_j \otimes x_k \otimes x_\ell + x_j \otimes x_k \otimes x_\ell \otimes x_m + x_k \otimes x_\ell \otimes x_m \otimes x_i + x_\ell \otimes x_m \otimes x_i \otimes x_j + x_m \otimes x_i \otimes x_j \otimes x_k \end{array} \right\rangle}$$

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where $i, j, k, \ell, m \in [n]$.

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In fact $V[n] \simeq H^1(\overline{\mathcal{M}_{0,n}}(\mathbb{R}); \mathbb{Q})$

This is a result of Etingof, Henriques, Kamnitzer, and Rains

They give a similar presentation for every H^p

More generally, classical representation theory of a finite group

abstract symmetries $\rightsquigarrow$ concrete linear symmetries

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A "symmetry" is an invertible self-transformation so you could argue that the notion of "transformation" is more fundamental than that of "symmetry."

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abstract symmetries $\rightsquigarrow$ concrete linear symmetries

Representation theory of a category

abstract transformations $\rightsquigarrow$ concrete linear transformations

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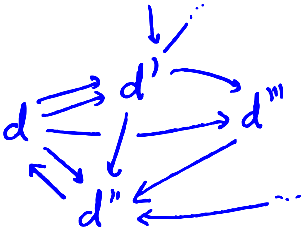

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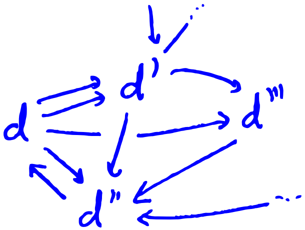

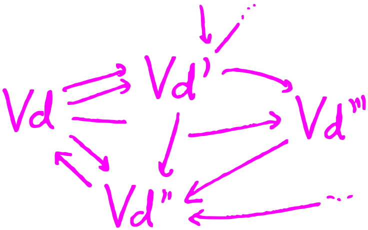

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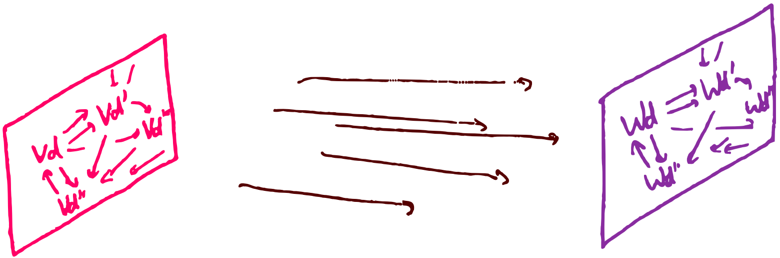

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<u>Defn</u> A representation V of a category $\mathcal{D}$ is generated by vectors $v_\alpha \in V d_\alpha$ if the smallest subrepresentation of V containing every v_α is V itself.

So generation uses linear combinations from $\mathbb{F}$ and arrows from $\mathcal{D}$.

If $\mathcal{D}$ has infinitely many arrows, a single vector could hypotheticaly generate an infinite-dimensional representation.

Taking $\mathcal{D}$ = one object $*$

$$\mathrm{Hom}_{\mathcal{D}}(*, *) = G$$

for some finite group recovers usual representation theory of a finite group.

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for some finite group recovers usual representation theory of a finite group. Allowing $\mathcal{D}$ to be enriched in $\mathbb{F}$-vector spaces recovers representation theory of an algebra

3

If you've got objects X_0, X_1, X_2, ...

3

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We want to Upgrade Indices!

(This process is also called "categorification")

<u>Example</u> Let M be a smooth manifold $\dim \geq 2$

$$X_n = \mathrm{Conf}_n(M) = \left\{ (x_1, x_2, \ldots, x_n) \in \prod_n M \;\middle|\; x_i \neq x_j \right\}$$

"configuration space of n distinct points in M"

__Example__ Let M be a smooth manifold dim ≥ 2

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We show how to build a representation of the category FI = objects $[n]$

arrows are injective functions

$$
\begin{tikzcd}
{[n]} \arrow[d, hook] & & X_{[n]} & & H^k(X_{[n]}\,;\,\mathbb{F}) \arrow[d] \\
{[m]} \arrow[rr, phantom, "\rightsquigarrow"] & & X_{[m]} \arrow[u] \arrow[rr, phantom, "\rightsquigarrow"] & & H^k(X_{[m]}\,;\,\mathbb{F})
\end{tikzcd}
$$

$$\begin{tikzcd} {[n]} \arrow[d, hook] & & X_{[n]} & & H^k(X_{[n]};\mathbb{F}) \arrow[d] \\ {[m]} & \rightsquigarrow & X_{[m]} \arrow[u] & \rightsquigarrow & H^k(X_{[m]};\mathbb{F}) \end{tikzcd}$$

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\begin{tikzcd}
{[3]} \arrow[d, hook] & = \{ & 1, \arrow[d, mapsto] & 2, \arrow[drr, mapsto] & 3 \} \arrow[drr, mapsto] & & \\
{[5]} & = \{ & 1, & 2, & 3, & 4, & 5 \}
\end{tikzcd}
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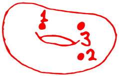
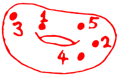
\rightsquigarrow
$$



$\in X_{[3]}$

$\uparrow$



$\in X_{[5]}$

$$\begin{array}{ccccc} [n] & & X_{[n]} & & H^k(X_{[n]};\mathbb{F}) \\ \big\downarrow & \rightsquigarrow & \big\uparrow & \rightsquigarrow & \big\downarrow \\ [m] & & X_{[m]} & & H^k(X_{[m]};\mathbb{F}) \end{array}$$

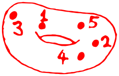
Let's draw this when $M =$ 

$$[3] = \{1, 2, 3\} \hookrightarrow [5] = \{1, 2, 3, 4, 5\}, \quad 1 \mapsto 1,\ 2 \mapsto 4,\ 3 \mapsto 5$$

$\rightsquigarrow$

$\in X_{[3]}$

$\uparrow$



$\in X_{[5]}$

$\rightsquigarrow$

$$\begin{array}{c} H^k(X_{[3]};\mathbb{F}) \\ \big\downarrow \\ H^k(X_{[5]};\mathbb{F}) \end{array}$$

So $\quad [n] \longmapsto H^k(X_{[n]}; \mathbb{F})$ is a representation of the category $\mathcal{FI}$.

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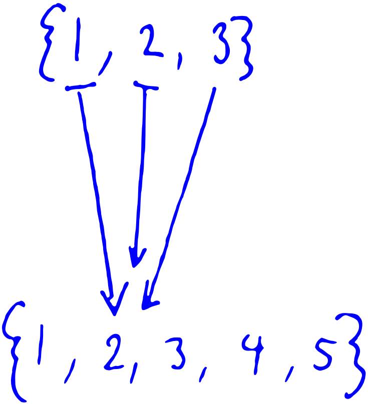
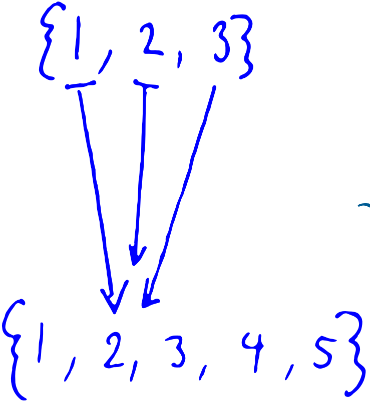
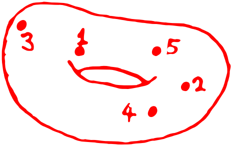
What would it mean for this representation to be finitely generated?

Suppose that M admits a nowhere-vanishing smooth vector field.

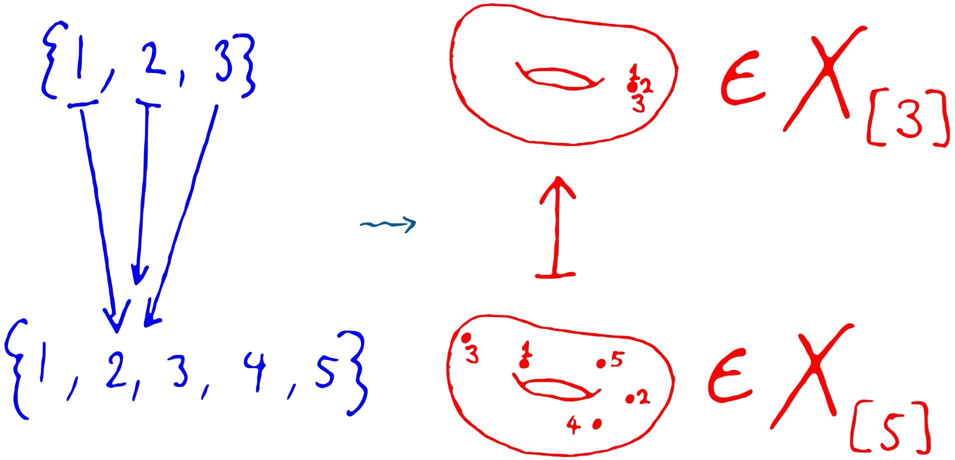

{1, 2, 3}
{1, 2, 3, 4, 5}

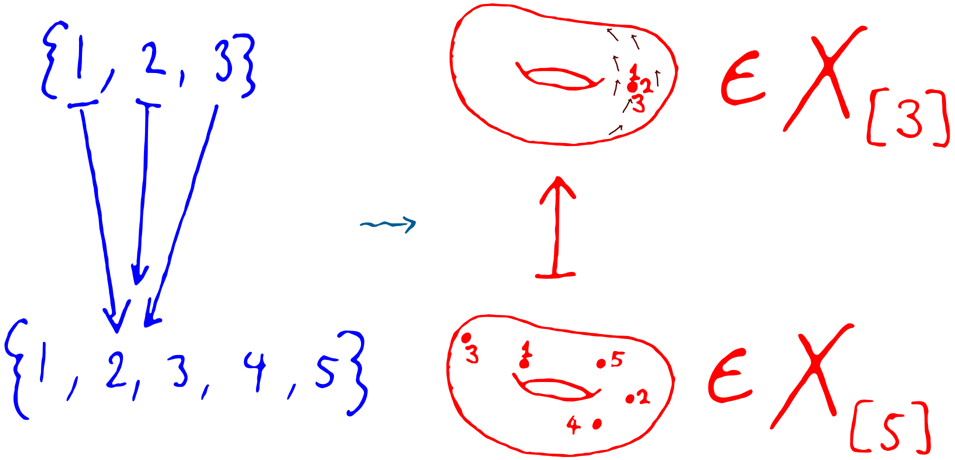


$\rightsquigarrow$



$\in X_{[5]}$


{1, 2, 3}
{1, 2, 3, 4, 5}
∈ X[3]
∈ X[5]
1 2 3
3 1 5 2 4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1
2
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1
2
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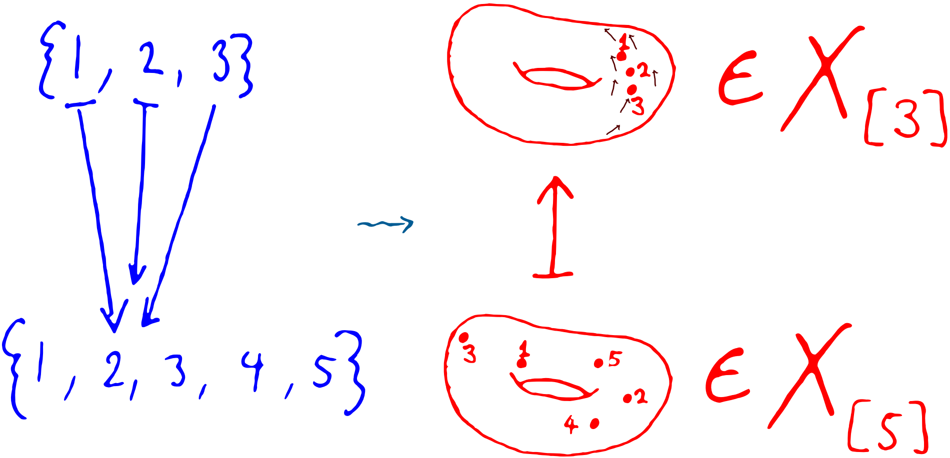
{1, 2, 3}
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$\in X_{[3]}$
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1
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Suppose that M admits a nowhere-vanishing smooth vector field. In this case we get a representation of Δ, the subcategory of $\mathcal{F}$ where the maps are required to be (weakly) monotone

4

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The most basic question about a representation is its dimension at an object d, $\dim V_d$.

More generally, we want to understand "multiplicity functions" $\chi : \{\text{representations}\} \longrightarrow \mathbb{Z}$ so that for any short exact sequence of representations

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$\chi(B) = \chi(A) + \chi(C)$.

A theory is judged by its helpfulness in computing multiplicities.

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a) $\mathcal{D}$ is Hom-finite: $\mathrm{Hom}_{\mathcal{D}}(d, d')$ is always a finite set

b) V is pointwise finite dimensional: V_d is always finite dimensional over $\mathbb{F}$.

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Artinian

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Multiplicity functions are "rigid"

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	Representations have predictable "eventual" behavior.	

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Representations are predictable from a finite initial segment.

Computationally similar to Gaussian elimination over a field.

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However, Sam and Snowden give a useful sufficient condition generalizing usual Gröbner theory. When $\mathcal{D}$ is "quasi-Gröbner" they can say a lot about multiplicity functions.

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The machine of Sam and Snowden works for several other important categories: $\mathcal{F}$, $\mathcal{F}^{op}$, $\mathcal{VI}$, $\mathcal{VA}$, $\mathcal{FI}_d$, ...

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The machine of Sam and Snowden works for several other important categories: $\mathcal{F}$, $\mathcal{F}^{op}$, $\mathcal{VI}$, $\mathcal{VA}$, $\mathcal{FI}_d$, ..

But there are other important categories (some known to be Noetherian) that seem not to admit Gröbner theory

5

At this point it would be reasonable to guess that $\mathcal{D}$ is Artinian if and only if it is a disjoint union of finite categories.

5

At this point it would be reasonable to guess that $\mathcal{D}$ is Artinian if and only if it is a disjoint union of finite categories. But it turns out there is an extremely famous (legitimately infinite) category whose representation theory is Artinian by a classical theorem.

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Ok, but Δ and $\mathcal{F}$ are pretty similar. (Some people even prefer $\mathcal{F}$!) Could $\mathcal{F}$ also be Artinian? If so, what is the analog of a cochain complex?

⑥

Presenting a practical combinatorial criterion to check if $\mathcal{D}$ is Artinian.

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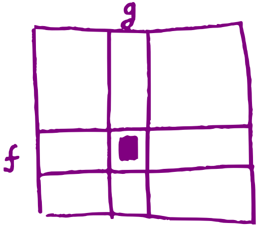

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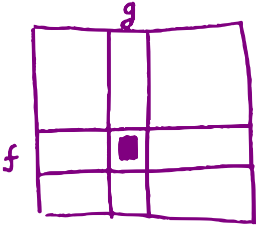

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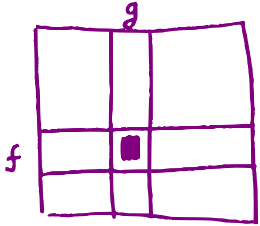

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If

$$\begin{tikzcd} & & d \arrow[dll, "f"'] \arrow[drr, "g"] & & \\ s : x \arrow[rr] & & y \arrow[rr] & & x \end{tikzcd}$$

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For each $d \in \mathcal{D}$, the relation $\underset{d}{\leq}$ is reflexive and transitive and so forms a preorder on the objects of $\mathcal{D}$.

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We think of $\mu(d)$ as a "joint maximum" for the preorder $\leq_d$.

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<u>Thm</u> $\mathcal{F}$ is Artinian over any field $\mathbb{F}$.

⑦

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It's mild combinatorics to provide a similar construction for every statement $[m] \underset{[n]}{\leq} [n+1]$.

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Pf Classify irreducible representations of $\mathcal{F}$. The theorem holds for them, and follows in general.

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Thank You! The paper can be found on my website.